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ADM, BONDI MASS, AND ENERGY CONSERVATION IN TWO-DIMENSIONAL DILATON GRAVITIES

Won Tae Kim^(a) and Julian Lee^(b)

*Center for Theoretical Physics and Department of Physics,
Seoul National University, Seoul 151-742, Korea*

ABSTRACT

We show how a stress-energy pseudotensor can be constructed in two-dimensional dilatonic gravity theories (classical, CGHS and RST) and derive the expression for the ADM mass in these theories from it. We define the Bondi mass for these theories by using the pseudotensor formalism. The resulting expression is the generalization of the expression for the ADM mass. The boundary condition needed for the energy conservation is also investigated. It is shown that under appropriate boundary conditions, our definition of the Bondi mass is exactly the ADM mass minus the matter radiation energy at null infinity.

^(a) E-mail address : wtkim@phyb.snu.ac.kr

^(b) E-mail address : lee@phyb.snu.ac.kr

Recently there has been lots of interests in two-dimensional dilaton gravity theories [1-5]. The main motivation comes from the fact that although they are two-dimensional toy models, they possess most of the interesting properties of the four-dimensional gravity theories such as the existence of the black hole solutions, Hawking radiations, etc. At the same time, they are more amenable to quantum treatments than their four-dimensional counterparts. Thus, it is hoped that they will pave a way to address interesting issues in quantum gravity such as information loss problem [6] in black hole physics.

It is a well known fact that one cannot construct a conserved stress-energy tensor in general relativity except for space-times having particular symmetries [7]. The fact that the stress-energy tensor for the matter fields alone is not conserved is not surprising since they exchange energies and momenta with the gravitational field. Furthermore, there is no notion corresponding to the stress-energy of the gravitational field which is a generally covariant tensor. However, we can introduce the concept of stress-energy for the gravitational field if we take the view that the general relativity can be treated as a spin-2 field theory in Minkowski background. The stress-energy so constructed will be a pseudotensor, in the sense they are not generally covariant, but is Lorentz covariant with respect to the background Minkowski metric. Just as in the case of four-dimensional Einstein gravity, we can show that the pseudotensor corresponding to energy density is a total derivative for the two-dimensional dilaton gravity theories. Therefore for a asymptotically flat space-time, the energy becomes a surface term defined at either spatial or null infinity. The former is the Arnowitt-Deser-Misner(ADM) mass [8] and the latter is called the Bondi mass [9]. The stress-energy pseudotensor is constructed in such a way that it is a conserved current. Then it is obvious that the difference of Bondi and ADM mass is integral of this current flowing out to null infinity. In the four-dimensional Einstein gravity, this current is interpreted as the flux of radiation energy. (It is to be understood that the word radiation refers to massless fields.) In the case of two-dimensional dilaton gravity theories, the graviton and dilaton fields have no propagating degrees of freedom and only the matter radiation is capable of escaping to null infinity.

In this paper we consider the energy conservation in two-dimensional dilaton grav-

ity theories, which are classical, Callan-Giddings-Harvey-Strominger(CGHS) [1] and Russo-Susskind-Thorlacius(RST) [2,3] models. We construct a stress-energy pseudotensor for these theories and rederive the well known expression for the ADM mass. We also specify the asymptotic boundary conditions needed for the total energy conservation. We then define the notion of Bondi mass by integrating the stress-energy pseudotensor along the null hypersurface instead of spatial one. The resulting expression is the generalization of the known expression for the ADM mass. It will be shown that our definition of Bondi mass is “correct” in the sense that it is exactly the ADM mass minus the integral of radiation energy flux at null infinity under reasonable boundary conditions. This result can be considered as the energy conservation applied to ADM and Bondi mass.

We consider dilaton gravity theories described by the action,

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[(e^{-2\phi} - \frac{\kappa}{2}\phi)R + 4e^{-2\phi}((\nabla\phi)^2 + \lambda^2) - \frac{1}{2}Q^2 R \frac{1}{\nabla^2} R - \frac{1}{2} \sum_{i=1}^N (\nabla f_i)^2 \right] \quad (1)$$

where $\kappa = Q = 0$ corresponds to the classical action, $\kappa = 0$, $2Q^2 = \frac{N}{12}$ gives the CGHS model [1], and $\kappa = 2Q^2 = \frac{N-24}{12}$ corresponds to RST model [2,3]. The latter two are the one-loop effective action incorporating the effect of trace anomaly of matter fields.

It is convenient to split the action above as

$$S = S_{DG} + S_{matter} \quad (2)$$

where

$$\begin{aligned} S_{DG} &\equiv \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\phi} \left[R + 4(\nabla\phi)^2 + 4\lambda^2 \right], \\ S_{matter} &\equiv \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[-\frac{1}{2} \sum_{i=1}^N (\nabla f_i)^2 - \frac{\kappa}{2}\phi R - \frac{1}{2}Q^2 R \frac{1}{\nabla^2} R \right]. \end{aligned} \quad (3)$$

S_{DG} governs the dynamics of the dilaton-gravity fields which is treated classically, whereas S_{matter} governs the classical dynamics of the scalar fields f_i and their possible one-loop effects in the large N limit [1,2,3]. Next, one expands the graviton-dilaton fields around the linear dilaton vacuum(LDV),

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \phi = -\lambda x^\alpha \eta_{\alpha\beta} \epsilon^\beta + \psi \quad (4)$$

where $\eta_{11} = -\eta_{00} = 1$, $\eta_{01} = \eta_{10} = 0$ and ϵ^α satisfy $\epsilon^\alpha \eta_{\alpha\beta} \epsilon^\beta = 1$. Note that the Poincaré symmetry present in the original Lagrangian is spontaneously broken by the vector ϵ^α . We now have a preferred coordinates ($x^0 = t$, $x^1 = q$) such that $x^\alpha \eta_{\alpha\beta} \epsilon^\beta = q$, which we will take from now on. Therefore, in contrast to the case of four-dimensional Einstein gravity where one has the energy-momentum four-vector for the whole system, only the concept of energy is meaningful in the case of two-dimensional dilaton gravities, which is usually called mass. Another consequence of this broken symmetry is the fact that the stress-energy pseudotensor constructed by the generalized Belinfante procedure [10] does not give any reasonable expression for the ADM mass [11] when integrated over a spatial hypersurface approaching spatial infinity.

To construct the stress-energy pseudotensor, one linearizes the equation of motion for the dilaton-graviton fields given by

$$G_{\mu\nu} = T_{\mu\nu} \quad (5)$$

to get [12]

$$G_{\mu\nu}^{(1)} = T_{\mu\nu} - G_{\mu\nu}^{(2)} \quad (6)$$

where

$$\begin{aligned} G_{\mu\nu} &\equiv 2\pi \frac{1}{\sqrt{-g}} \frac{\delta S_{DG}}{\delta g^{\mu\nu}} = 2e^{-2\phi} \left[\nabla_\mu \nabla_\nu \phi + g_{\mu\nu} ((\nabla\phi)^2 - \nabla^2 \phi - \lambda^2) \right], \\ T_{\mu\nu} &\equiv -2\pi \frac{1}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}, \end{aligned} \quad (7)$$

and $G_{\mu\nu}^{(1)}$ is the part of $G_{\mu\nu}$ linear in $h_{\mu\nu}$ and ψ , and $G_{\mu\nu}^{(2)}$ is the part which is of second or higher order. Then the second term on the right-hand side of Eq. (6) can be interpreted as a stress-energy of the dilaton-graviton fields. The energy current,

$$j^\mu \equiv T^{\mu\nu} - G^{(2)\mu\nu} \quad (8)$$

satisfies the conservation law, where it is to be understood that the Minkowski metric is used in the lowering and upping of indices. To see this, we note that $G^{(1)\mu 0} \equiv \eta^{\mu\alpha} \eta^{0\beta} G_{\alpha\beta}^{(1)}$ is in the form which is identically conserved. This fact follows from the so called linearized Bianchi identity, which in turn follows from the gauge invariance. The

gauge transformation corresponding to a diffeomorphism is given by

$$\begin{aligned}\delta g^{\mu\nu} &= \epsilon^\alpha \partial_\alpha g^{\mu\nu} - (\partial_\alpha \epsilon^\mu) g^{\alpha\nu} - (\partial_\alpha \epsilon^\nu) g^{\mu\alpha}, \\ \delta \phi &= \epsilon^\lambda \partial_\lambda \phi, \\ \delta f_i &= \epsilon^\lambda \partial_\lambda f_i\end{aligned}\tag{9}$$

where $\epsilon(x)$ is an arbitrary function parameterizing the gauge transformation. We note that not only the whole action is invariant under the transformation Eq. (9), but also the dilaton-gravity part of the action S_{DG} is gauge invariant. From this fact we get the Bianchi identity:

$$(\partial_\lambda g^{\mu\nu}) \frac{\delta S_{DG}}{\delta g^{\mu\nu}} + 2\partial_\alpha \left(g^{\alpha\nu} \frac{\delta S_{DG}}{\delta g^{\lambda\nu}} \right) + (\partial_\lambda \phi) \frac{\delta S_{DG}}{\delta \phi} = 0.\tag{10}$$

Since Eq. (10) is an identity, it will hold order by order when we expand it in terms of $h_{\mu\nu}$ and ψ . To linear order, one has

$$2\partial_\alpha \eta^{\alpha\nu} G_{\lambda\nu}^{(1)} - \lambda \eta_{\lambda\alpha} \epsilon^\alpha F^{(1)} = 0\tag{11}$$

where $F^{(1)}$ are the parts linear in $h_{\mu\nu}$ and ψ of $\frac{\delta S}{\delta \phi}$. Eq. (11) is called the linearized Bianchi identity. In the (t, q) coordinate given above, the linearized Bianchi identity (11) gives

$$\partial_\mu G^{(1)\mu 0} \equiv 0.\tag{12}$$

Showing that $G^{(1)\mu 0}$ is a conserved current. Therefore we define $G^{(1)\mu\nu}$, or alternatively $T^{(1)\mu\nu} - G^{(2)\mu\nu}$, as the stress-energy pseudotensor.

Although j^μ satisfies the local conservation law due to Eq. (6) and Eq. (12), it does not follow immediately that the total energy

$$E_{ADM} \equiv \int_{-\infty}^{\infty} dq j^0 = \int_{-\infty}^{\infty} dq G^{(1)00}\tag{13}$$

is conserved without specifying the boundary conditions. However, if we consider a space-time which approaches the LDV at spatial infinity fast enough, then

$$j^1 = T^{10} - G^{(2)10}\tag{14}$$

would vanish at $q \rightarrow \pm\infty$ and the total energy E_{ADM} becomes time-independent. On the other hand $G^{(1)\mu 1}$, which would correspond to the momentum current, does not even

satisfy the local conservation law since the translational symmetry in spatial direction is spontaneously broken by the LDV. Using the expression at the end of Eq. (13), one gets after some straightforward algebra

$$E_{ADM} = 2e^{2\lambda q}(\partial_q \psi + \lambda \frac{h_{11}}{2})|_{q \rightarrow \infty}. \quad (15)$$

It is easy to see that the contribution from $q \rightarrow -\infty$ vanishes. Writing $g_{11} = e^{2\rho}$, we get the expression

$$E_{ADM} = 2e^{2\lambda q}(\partial_q \psi + \lambda \rho)|_{q \rightarrow \infty} \quad (16)$$

if ρ falls to zero at $q \rightarrow \infty$ not slower than $e^{-2\lambda q}$. Eq. (16) is the expression used often in the literature [13,14]. Although the conformal gauge is commonly used in order to get this expression, note that we only imposed the condition that the dilaton-graviton fields should approach LDV fast enough so that the current j^1 given by Eq. (14) should vanish as $q \rightarrow \pm\infty$. We see that as long as the dilaton-gravity fields approach LDV configuration and the classical matter fields f_i vanish at $q \rightarrow \pm\infty$ the first term T^{10} as a whole vanishes. Also, since $G^{(2)\mu\nu}$ consists of terms quadratic or higher order in ρ, ψ times the common factor $e^{-2\phi}$ which behaves as $e^{2\lambda q}$ at $q \rightarrow \pm\infty$, we easily see that it vanishes as $q \rightarrow -\infty$. The only nontrivial condition is given for $q \rightarrow \infty$, since the exponential factor blows up. It is sufficient to require that

$$h_{\mu\nu} \sim e^{-2\lambda q}, \quad \psi \sim e^{-2\lambda q} \quad (17)$$

as $q \rightarrow \infty$. Then the expression (15), or equivalently (16), gives the ADM mass which is time-independent. (Naively one might be tempted just to impose the condition $e^{\lambda q} h_{\mu\nu} \rightarrow 0$, $e^{\lambda q} \psi \rightarrow 0$ at $q \rightarrow \infty$ so that $G^{(2)\mu\nu}$ vanishes. However, this condition would not be interesting since if ρ and ψ vanish faster than $e^{-\lambda q}$ but slower than $e^{-2\lambda q}$ this configuration is a physically pathological case since the ADM mass blows up.)

Now let us introduce the light-front coordinates $\sigma_{\pm} \equiv t \pm q$. The Bondi mass $B(\sigma^-)$ can then be defined in a straightforward manner by integrating the energy flux along the null line $t - q = \sigma^-$, *i.e.*

$$\begin{aligned} B(\sigma^-) &\equiv \frac{1}{2} \int_{-\infty}^{\infty} d\sigma^+ j^-(\sigma^+, \sigma^-) \\ &= \frac{1}{2} \int_{-\infty}^{\infty} d\sigma^+ G^{(1)-0}(\sigma^+, \sigma^-) \end{aligned}$$

$$\begin{aligned}
&= 2e^{2\lambda q}(\partial_q \psi + \lambda \rho)|_{\sigma^+ \rightarrow \infty} \\
&= 2e^{\lambda(\sigma^+ - \sigma^-)}((\partial_+ - \partial_-)\psi + \lambda \rho)|_{\sigma^+ \rightarrow \infty}
\end{aligned} \tag{18}$$

where $G^{(1)-0} \equiv G^{(1)00} - G^{(1)10}$ and the second line of Eq. (18) follows from Eq. (6). Note that we have

$$\lim_{\sigma^- \rightarrow -\infty} B(\sigma^-) = E_{ADM}.$$

Next, we will investigate the boundary condition required at the null infinity so that one can interpret $B(\sigma^-)$ as the energy left in the system after the radiation has been emitted. Before that, we will compare the results of this section with other expressions of ADM and Bondi mass found in the literature and show they agree. The following expression for the ADM mass

$$E_{ADM} = 2e^{2\lambda q}(\partial_q + \lambda)(\psi - \psi^2)_{q \rightarrow \infty} \tag{19}$$

was obtained by Bilal and Kogan [15] using the Hamiltonian formalism. They required the boundary conditions at spatial infinity,

$$\begin{aligned}
\psi &\sim e^{-\lambda q}, & \rho &\sim e^{-\lambda q}, \\
e^{2\lambda q}(\psi - \rho) &\rightarrow 0.
\end{aligned} \tag{20}$$

By imposing the conditions (17) and (20) simultaneously, *i.e.* requiring

$$\begin{aligned}
\psi &\sim e^{-2\lambda q}, & \rho &\sim e^{-2\lambda q}, \\
e^{2\lambda q}(\psi - \rho) &\rightarrow 0,
\end{aligned} \tag{21}$$

one can replace ρ in Eq. (16) by ψ and drop the ψ^2 in Eq. (19), giving the common expression

$$E_{ADM} = 2e^{2\lambda q}(\partial_q + \lambda)\psi|_{q \rightarrow \infty}. \tag{22}$$

A different form of stress-energy pseudotensor was obtained in Ref. [11] as a Noether current associated with the diffeomorphism invariance of the dilaton gravity Lagrangian. The ADM mass obtained from this pseudotensor is

$$E_{ADM} = -\frac{1}{2} \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu} \left[e^{-2\phi} \nabla_\mu e_\nu + \frac{1}{2} e_\mu f_i \nabla_\nu f_i \right] \tag{23}$$

where $e^0 = 1$ and $e^1 = 0$, and the upping and lowering of indices are done by using the full metric $g_{\alpha\beta}$. Expanding the metric and dilaton fields around the LDV configuration, we again get the expression which agrees with Eq. (16) under the boundary condition

$$e^{2\lambda q} h_{10} \rightarrow 0 \quad (24)$$

together with Eq. (17).

Also we note that in case of CGHS model, a different formula for Bondi mass was obtained in Ref. [1],

$$B(\sigma^-) = 2e^{\lambda(\sigma^+ - \sigma^-)}((\partial_+ - \partial_-)\psi + \lambda\rho) + \frac{N}{12}(\partial_- - \partial_+)\rho|_{\sigma^+ \rightarrow \infty}, \quad (25)$$

which is supposed to hold for general conformal coordinates. As was noted in Ref. [1], the second term vanishes in the coordinates where the dilaton-gravity fields approach the standard form of LDV $\rho = 0, \phi = -\frac{\lambda}{2}(\sigma^+ - \sigma^-)$ at $\sigma^+ \rightarrow \infty$, and in this case Eq. (25) reduces to Eq. (18).

Now we consider the difference between E_{ADM} and $B(\sigma^-)$ and investigate the asymptotic boundary conditions under which it equals the matter radiation energy emitted to the null infinity. Obviously, it can be represented by the integral of the current flux along the null line at infinity from the point $(\sigma^+, \sigma^-) = (\infty, -\infty)$ to the point (∞, σ^-) ,

$$\begin{aligned} E_{ADM} - B(\sigma^-) &= \lim_{\sigma^+ \rightarrow \infty} \frac{1}{2} \int_{-\infty}^{\sigma^-} d\sigma^- j^+(\sigma^+, \sigma^-) \\ &= \lim_{\sigma^+ \rightarrow \infty} \frac{1}{2} \int_{-\infty}^{\sigma^-} d\sigma^- G^{(1)+0}(\sigma^+, \sigma^-) \\ &= - \lim_{\sigma^+ \rightarrow \infty} \int_{-\infty}^{\sigma^-} d\sigma^- \partial_- \left[2e^{\lambda(\sigma^+ - \sigma^-)} ((\partial_+ - \partial_-)\psi + \lambda\rho) (\sigma^+, \sigma^-) \right] \end{aligned} \quad (26)$$

where the second line follows from Eq. (6), as before. Again, we want to find the boundary condition such that the contribution from $G^{(2)\mu\nu}$ to j^μ flux is negligible. We see that it is sufficient to require

$$\begin{aligned} \psi &\sim e^{-\lambda\sigma^+}, & \rho &\sim e^{-\lambda\sigma^+}, \\ e^{\lambda\sigma^+} h_{\pm\pm} &\rightarrow 0, \end{aligned} \quad (27)$$

as $\sigma^+ \rightarrow \infty$. Again, at $\sigma^+ \rightarrow -\infty$ it is enough to require that the configuration approaches the LDV. Indeed, for this boundary condition the total matter radiation energy flux at null infinity is given by

$$\begin{aligned}
& \int_{-\infty}^{\sigma^-} d\sigma^- \sqrt{-g} T^{+0}|_{\sigma^+ \rightarrow \infty} \\
& \equiv \frac{1}{2} \int_{-\infty}^{\sigma^-} d\sigma^- \sqrt{-g} (T^{++} + T^{+-})|_{\sigma^+ \rightarrow \infty} \\
& = \int_{-\infty}^{\sigma^-} d\sigma^- e^{-2(\rho+\phi)} (2\partial_-^2 \phi - 4\partial_- \rho \partial_- \phi - 2\partial_+ \partial_- \phi + 4\partial_+ \phi \partial_- \phi + \lambda^2 e^{2\rho})|_{\sigma^+ \rightarrow \infty} \\
& = \int_{-\infty}^{\sigma^-} d\sigma^- \partial_- \left[2e^{\lambda(\sigma^+ - \sigma^-)} ((\partial_+ - \partial_-)\psi + \lambda\rho) \right]_{\sigma^+ \rightarrow \infty} \tag{28}
\end{aligned}$$

where the third line follows from the equation of motion (5). Comparing Eq. (26) with Eq. (28), we see that

$$E_{ADM} - B(\sigma^-) = \lim_{\sigma^+ \rightarrow \infty} \int_{-\infty}^{\sigma^-} d\sigma^- \sqrt{-g(\sigma^+, \sigma^-)} T^{+0}(\sigma^+, \sigma^-). \tag{29}$$

Thus, we have shown that under the boundary condition (27), our definition of Bondi mass agrees with the notion that the Bondi mass is the energy left in the system after the radiation has been emitted to null infinity.

In this paper, we showed how the expression for the ADM mass can be derived using a stress-energy pseudotensor. We then defined the notion of Bondi mass which is consistent with the usual definition as the energy left in the system after the radiation has been emitted. We also investigated the boundary conditions required at spatial and null infinities in order for the energy conservation to hold. As in the case of four-dimensional Einstein gravity, these asymptotic boundary conditions were crucial in establishing the energy conservation. The fact that the energy conservation is violated when these boundary conditions are not satisfied simply tells us that we cannot treat the system as being “isolated” one living on the Minkowski background anymore. The concept of energy is meaningless in this case.

Another attempt of constructing Bondi mass can be found in Ref. [16]. However, we are puzzled by the fact it violates the energy conservation even for reasonable boundary conditions such as in the case of a evaporating black hole in RST model. On the other hand, it is claimed in Ref. [16] that the Bondi mass is positive definite, whereas our Bondi mass is not necessarily so. We believe our definition of Bondi mass is more

sensible since it satisfies the energy conservation. By applying our definition of the Bondi mass to the case of the evaporating black hole in RST model, we get the correct energy conservation in variance with the results of Ref. [16]. The details can be found in Ref. [17].

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